

Second-Order Finite Elements for Deformable Surfaces

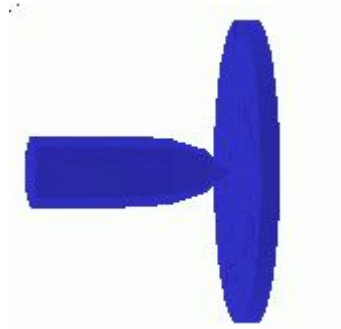
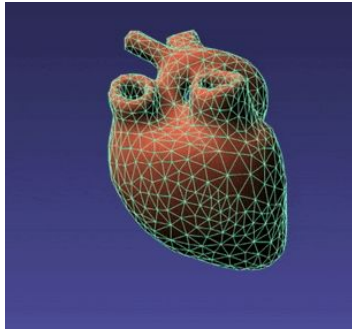
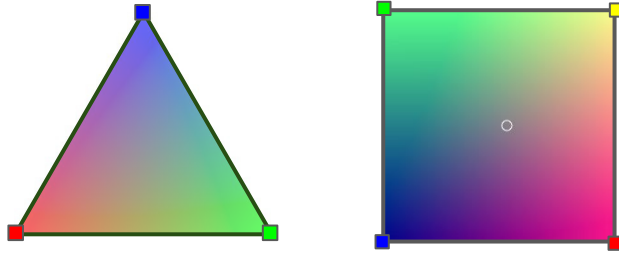
Qiqin Le, Yitong Deng, Jiamu Bu, Bo Zho, Tao Du



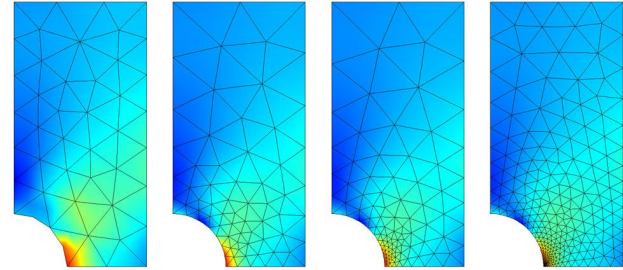
High Order Elements

Finite Element Method

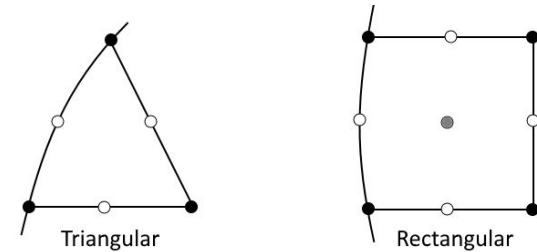
Finite Element



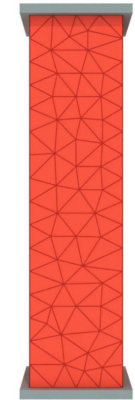
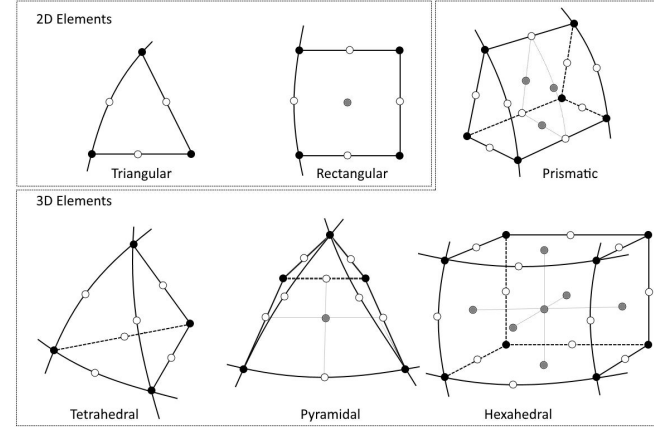
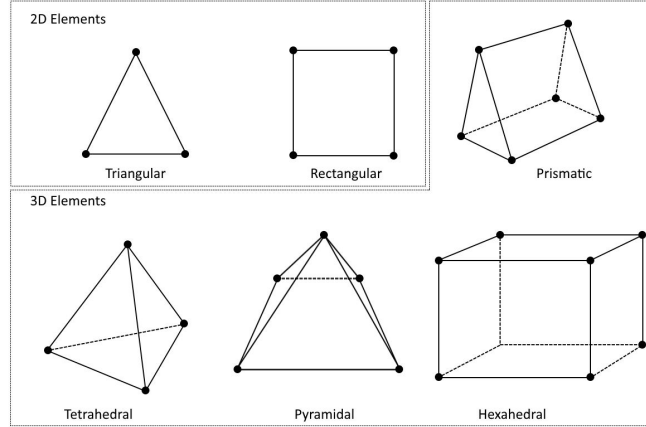
h-refinement / mesh quality



p-refinement / high order element



Some Elements



Coarse Mesh



P1 Coarse



P1 Refined

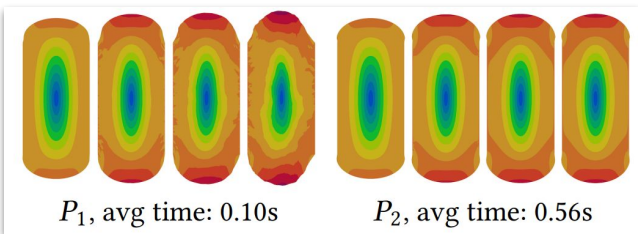
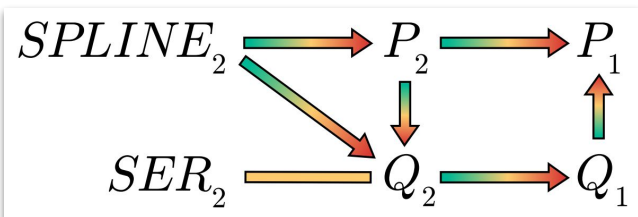


P2



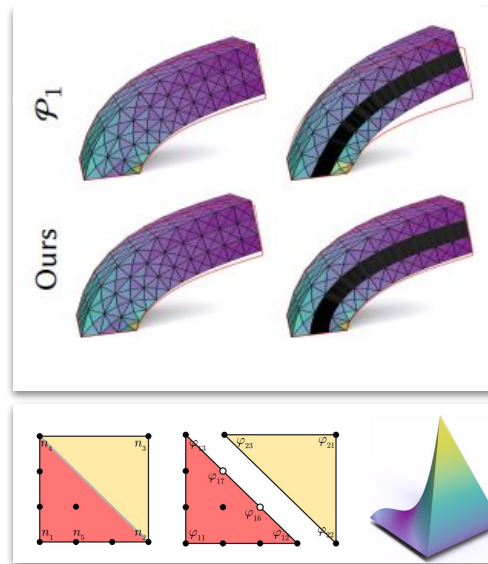
P3

How to reduce discretisation error ?



2023 - A Large-Scale Comparison of Tetrahedral and Hexahedral Elements for Solving Elliptic PDEs with the Finite Element Method

P2 are more performant than P1
for the same target error

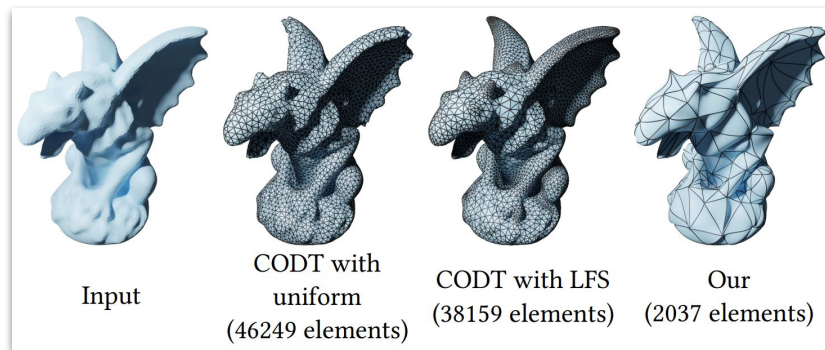
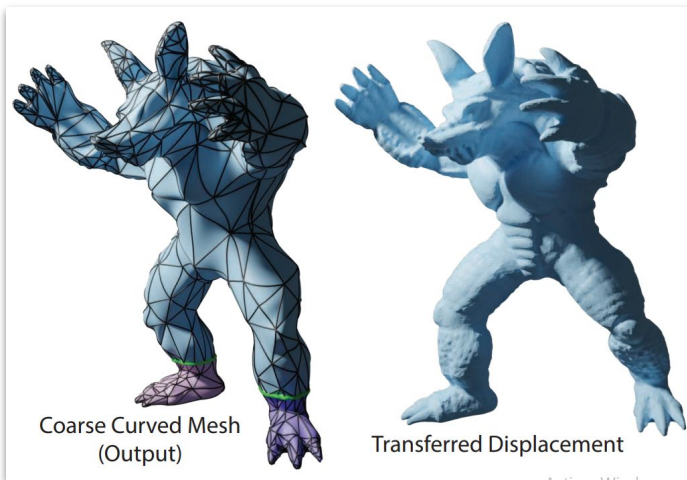


2018 - Decoupling Simulation Accuracy from Mesh Quality

High order elements don't care
about mesh quality

Coarse mesh with High Order Element

Zhongshi Jiang, Ziyi Zhang, Yixin Hu,
Teseo Schneider, Denis Zorin, Danielle Panozzo

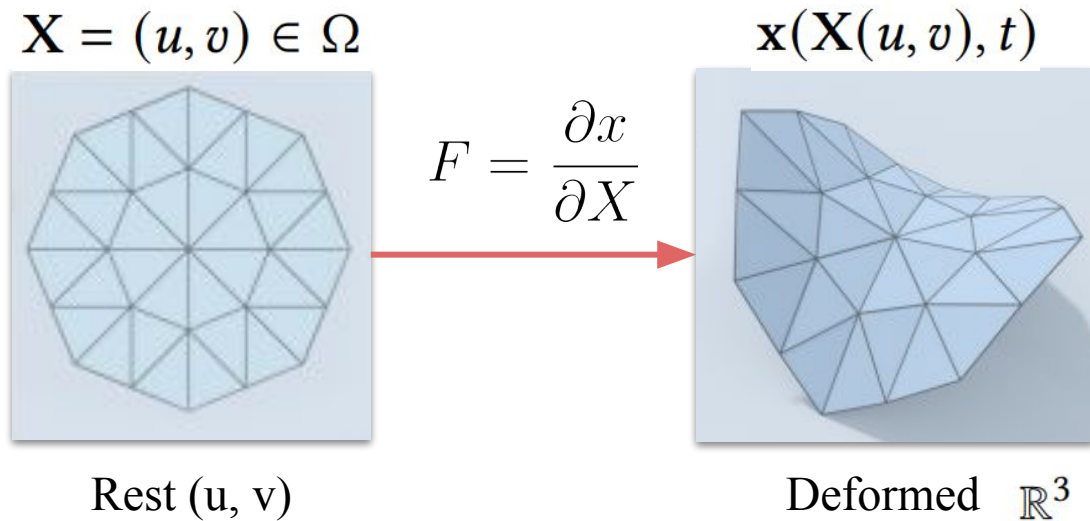


2021 - Bijective and Coarse High-Order Tetrahedral Meshes

High order element are way more efficient
in representing curved surface

Thin Shell Element

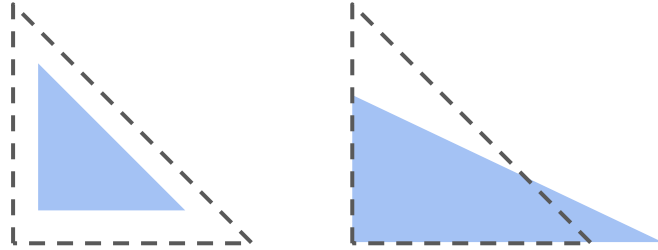
Thin Shell Element



$$\mathbf{F}_{\mathbf{x}(t)} : \Omega \rightarrow \mathbb{R}^{3 \times 2}$$

Energies

Stretch and Shear

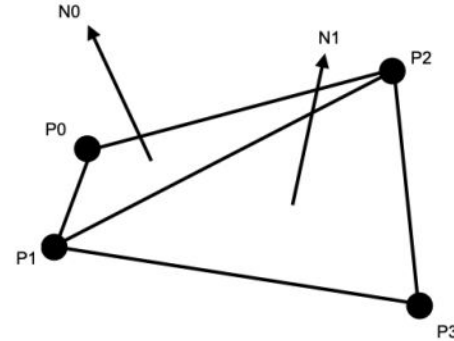


$$\varepsilon_{hyper} = \frac{1}{2}(F^T F - I)$$

$$\Psi = \frac{\lambda}{2}(\text{tr} \varepsilon)^2 + \mu \text{tr}(\varepsilon^2)$$

$$E_{ss} = \int_{\Omega} \Psi_{ss}(u, v) du dv$$

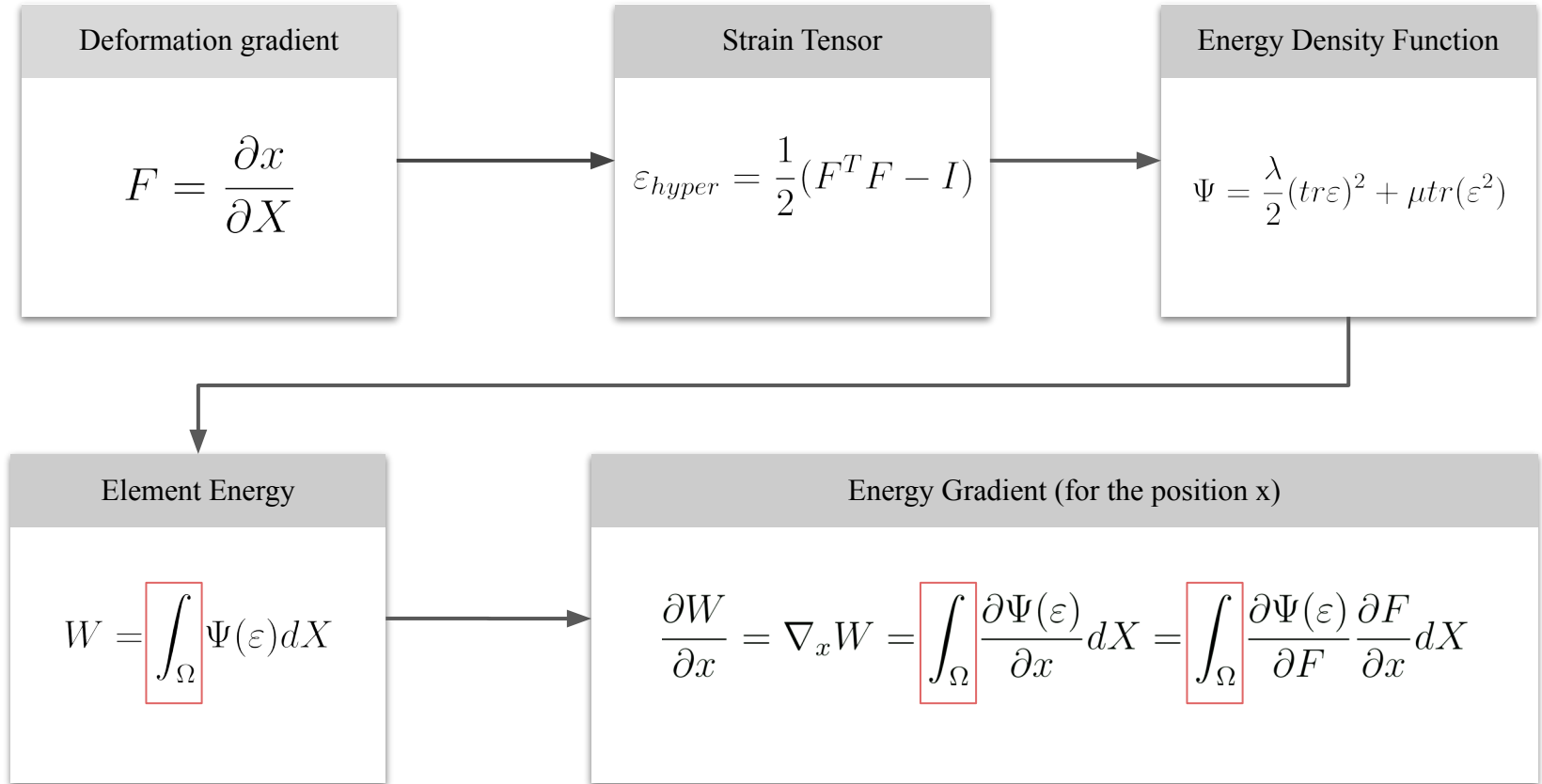
Bending



$$\Psi_{bd}(\mathbf{X}(u, v)) = k H_{\mathbf{x}(t)}^2(u, v)$$

$$E_{bd} = \int_{\Omega} \Psi_{bd}(u, v) du dv$$

Stretch and Shear

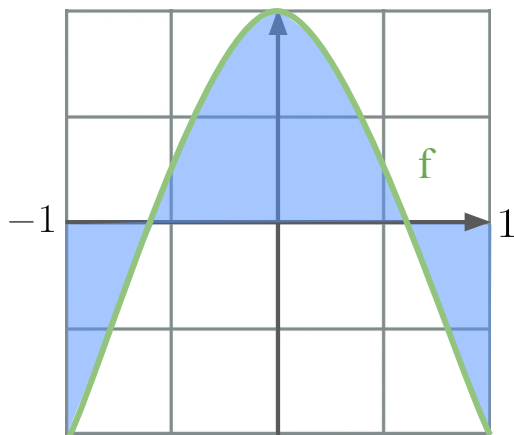


Gauss–Legendre Quadrature

$$\int_{-1}^1 f(x) dx \approx \sum_{i=1}^n w_i f(x_i)$$

Accurate approximation
if $f(x)$ is well-approximated by a polynomial of
degree $2n - 1$ or less on $[-1, 1]$

$$f(x) = -2x^2 + 1$$



$$\int_{-1}^1 f(x) dx = \frac{2}{3} \approx 0.666667$$

Number of points, n	Points, x_i		Weights, w_i	
1	0		2	
2	$\pm \frac{1}{\sqrt{3}}$	$\pm 0.57735\dots$	1	
3	0		$\frac{8}{9}$	0.888889...
	$\pm \sqrt{\frac{3}{5}}$	$\pm 0.774597\dots$	$\frac{5}{9}$	0.555556...

$$n = 2 \quad w_1 = w_2 = 1 \quad x_1 = -\frac{1}{\sqrt{3}}, \quad x_2 = \frac{1}{\sqrt{3}}$$

$$\int_{-1}^1 f(x) dx \approx w_1 f(x_1) + w_2 f(x_2) = 0.666667$$

Two Dimensional Domains

$$\int_{-1}^1 \int_{-1}^1 f(x, y) dx dy \approx \sum_{i=1}^n \sum_{j=1}^n w_i w_j f(x_i, y_j)$$

Same integration values and weights !

$$f(x, y) = 3x^2 - y^2 + x - 3$$

$$n = 2 \quad w_1 = w_2 = 1$$

$$x_1 = -\frac{1}{\sqrt{3}}, \quad x_2 = \frac{1}{\sqrt{3}} \quad y_1 = -\frac{1}{\sqrt{3}}, \quad y_2 = \frac{1}{\sqrt{3}}$$

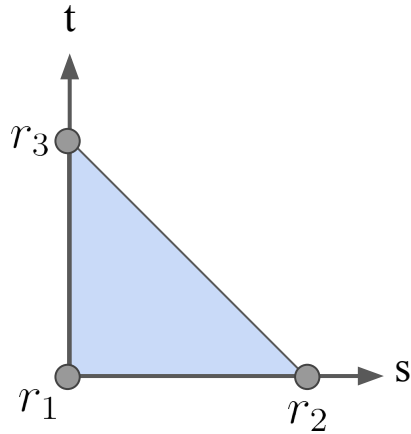
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	$\pm \sqrt{\frac{3}{5}}$	$\pm 0.774597\dots$	$\frac{5}{9}$	0.555556...

$$\int_{-1}^1 \int_{-1}^1 f(x, y) dx dy \approx \sum_{i=1}^2 \sum_{j=1}^2 w_i w_j f(x_i, y_i)$$

$$= w_1 w_1 f(x_1, y_1) + w_2 w_1 f(x_2, y_1) + w_1 w_2 f(x_1, y_2) + w_2 w_2 f(x_2, y_2)$$

Shape Function

Reference Triangle



Triangle Shape Functions

$$N_1(r) = 1 - s - t$$

$$N_2(r) = s$$

$$N_3(r) = t$$

$$\sum_{i=1}^n N_i(r) = 1$$

Partial Derivatives

$$\frac{\partial N_1}{\partial s} = -1 \quad \frac{\partial N_1}{\partial t} = -1$$

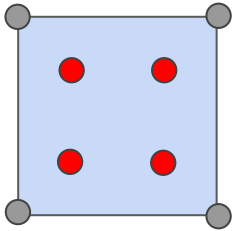
$$\frac{\partial N_2}{\partial s} = 1 \quad \frac{\partial N_2}{\partial t} = 0$$

$$\frac{\partial N_3}{\partial s} = 0 \quad \frac{\partial N_3}{\partial t} = 1$$

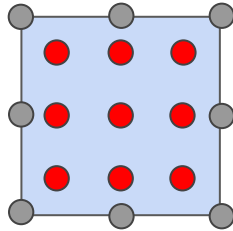
Constant = easy integration

Gauss–Legendre Quadrature for FEM

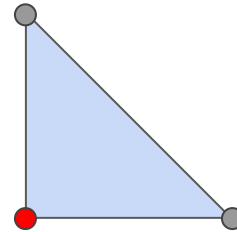
Linear Quad



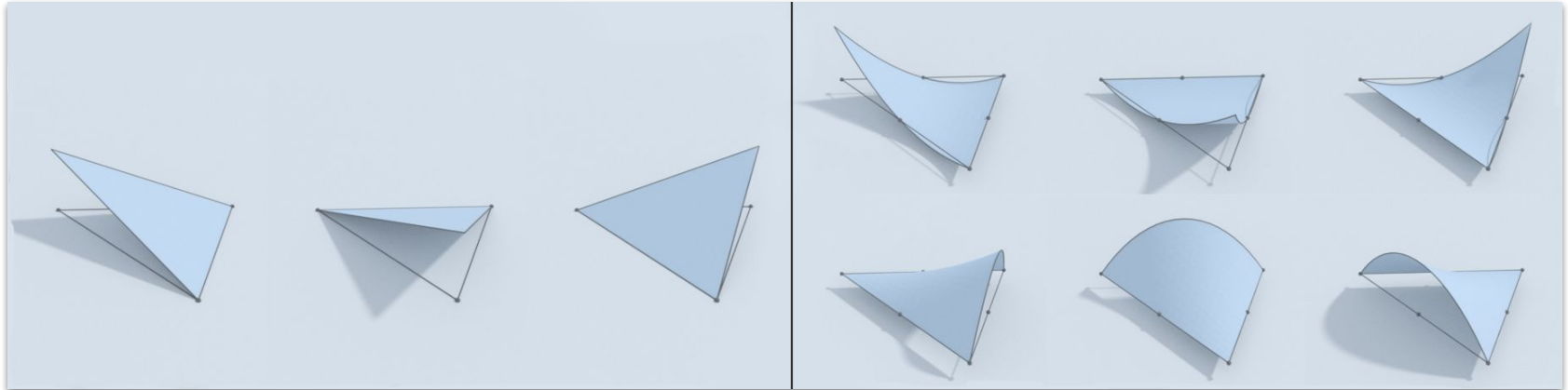
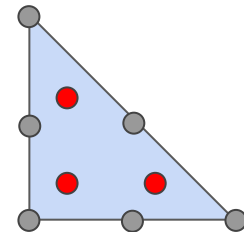
Quadratic Quad



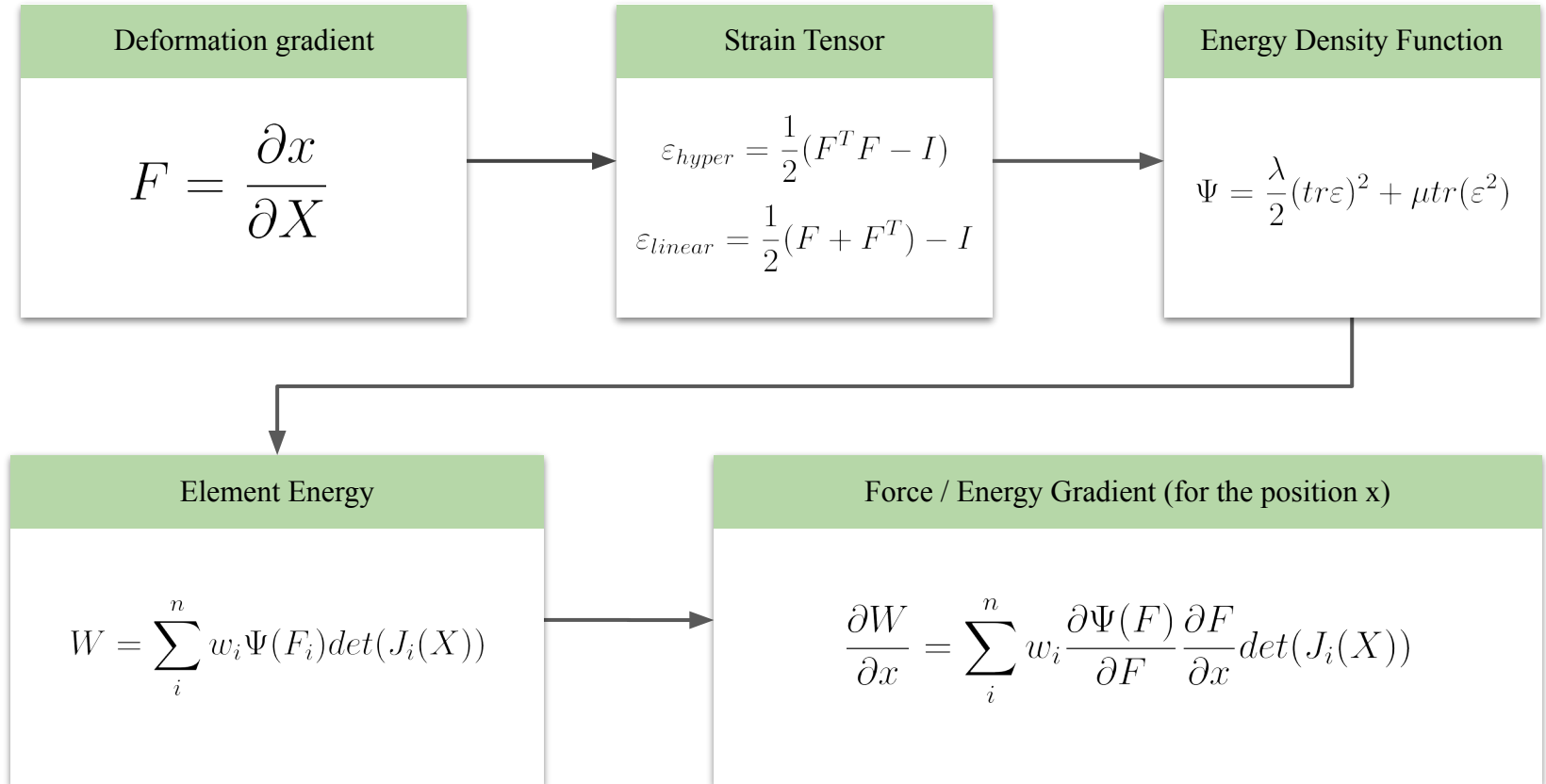
Linear Triangle



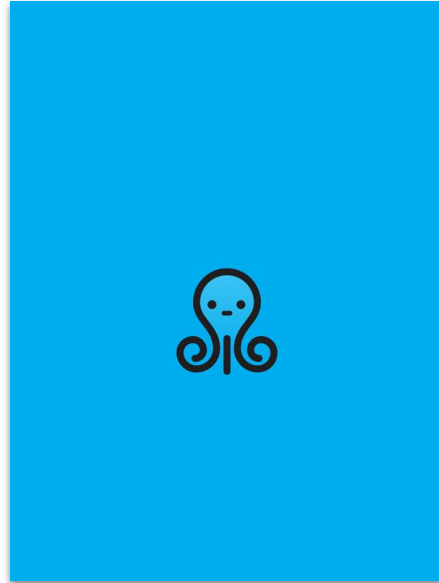
Quadratic Triangle



Stretch and Shear Forces



What is the bending energy ?



boire de l'eau

Mean Curvature

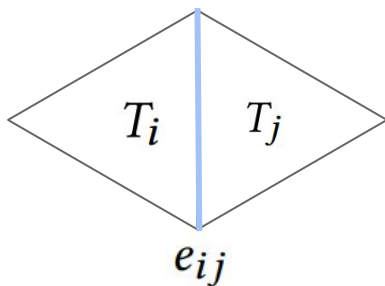
Restricted Delaunay Triangulations and Normal Cycle - Cohen-Steiner and Morvan [2003]

$$\alpha_{\Pi}(B) = \sum_{\{T_j\}} \int_{B \cap \Pi(T_j)} H_{\Pi}(\mathbf{p}) d\mathbf{p} + \sum_{\{e_{ij}\}} A(B \cap \Pi(e_{ij}))$$

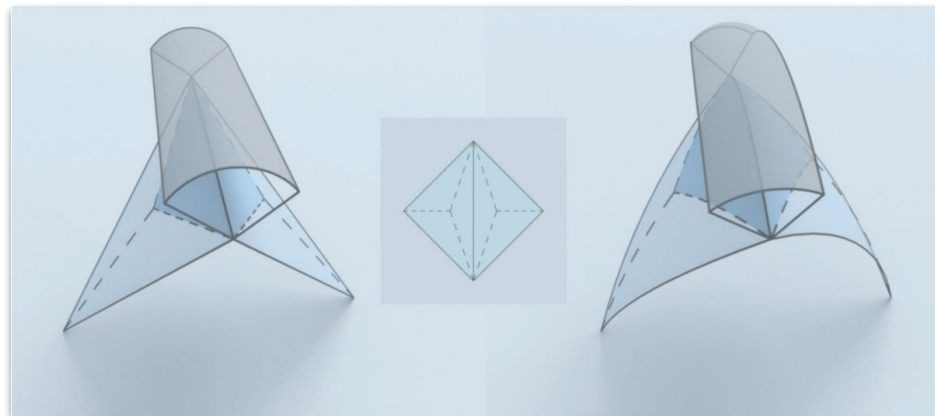
Π surface

$B \subset \mathbb{R}^3$ (Borel) set

$H_{\Pi}(\mathbf{p})$ mean curvature
at a surface point $\mathbf{p}$



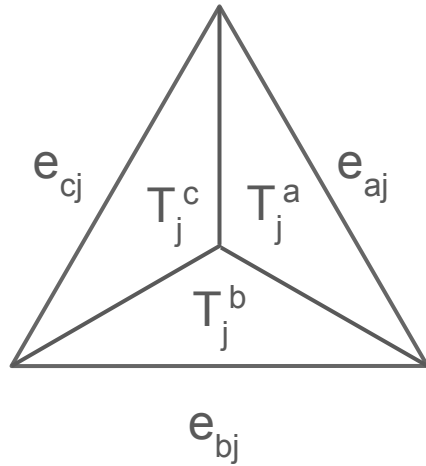
"tube formula" = $A(B \cap \{\Pi(e_{ij})\})$



if convex $A > 0$

if concave $A < 0$

Discretized Mean Curvature



$$f(u, v) \approx \hat{f}(u, v) = \sum_i f_i \phi_i(u, v), \quad \forall (u, v) \in \Omega,$$

$$I_\Omega[f] = \int_\Omega f \, du \, dv \approx \sum_{(\mathbf{q}_k, w_k)} w_k \hat{f}(\mathbf{q}_k),$$

Mean Curvature
at quadrature point

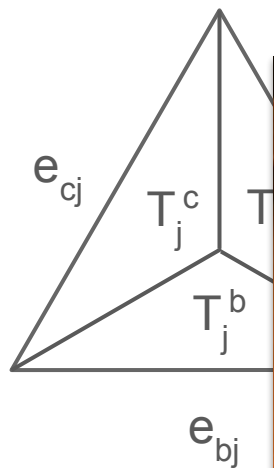
“tube formula”

$$\bar{H}_{\hat{\mathbf{x}}^{n+1}}(u, v) = H_{\hat{\mathbf{x}}^{n+1}}(u, v) + \frac{A(B_{ij} \cap \hat{\mathbf{x}}^{n+1}(e_{ij}))}{|\hat{\mathbf{x}}^{n+1}(T_i^{k(e_{ij})}) \cup \hat{\mathbf{x}}^{n+1}(T_j^{k(e_{ij})})|}$$

Discretized
mean curvature

Surface area of the two deformed
subdivided triangles (adjacent)

Discretized Mean Curvature



$$\bar{H}_{\hat{\mathbf{x}}^n}$$



subdivided triangles (adjacent)

$$\forall (u, v) \in \Omega,$$

$$(q_k),$$

ce''

$$|c(e_{ij})|$$

med

...

Restricted Delaunay Triangulations and Normal Cycle - Cohen-Steiner and Morvan [2003]



Vidéo